Journal of Logic, Language, and Information **8:** 83–110, 1999.

The Complexity of Modellability in Finite and Computable Signatures of a Constraint Logic for Head-Driven Phrase Structure Grammar

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(Received 9 April 1997; in final form 21 November 1997)

Abstract. The SRL (speciate re-entrant logic) of King (1989) is a sound, complete and decidable logic designed specifically to support formalisms for the HPSG (head-driven phrase structure grammar) of Pollard and Sag (1994). The SRL notion of modellability in a signature is particularly important for HPSG, and the present paper modifies an elegant method due to Blackburn and Spaan (1993) in order to prove that

- modellability in each computable signature is Π^0_1,
- modellability in some finite signature is Π^0_1-hard (hence not decidable), and
- modellability in some finite signature is decidable.

Since each finite signature is a computable signature, we conclude that Π^0_1-completeness is the least upper bound on the complexity of modellability both in finite signatures and in computable signatures, though not a lower bound in either.

Key words: Complexity theory, constraint logics, HPSG formalisms

1. Introduction

HPSG (head-driven phrase structure grammar) is one of several grammars created during the eighties partly in protest at the increasingly vague and indeterminate mathematical formalisms underlying the various forms of Chomskyan grammar. Pollard and Sag (1994) envisage a formalism for HPSG in which formal-syntactic entities called descriptions are closed under negation, conjunction and disjunction, each description denotes a set of linguistic objects, and the negation, conjunction and disjunction of the formal syntax denote respectively the complement, intersection and union of sets of linguistic objects. A theory is a set of descriptions, and the denotation of a theory is the intersection of the denotations of its members. The formalism expresses an account of a natural language as a theory that denotes the collection of linguistic objects that the mature users of the natural language deem grammatical. Pollard and Sag (1994) do not themselves provide an explicit formalism for HPSG, but rather cite a number of formal logics as candidate bases for such a formalism. Among these candidates is the SRL (speciate re-entrant logic) of King (1989), and Pollard (1998) develops a SRL-based formalism for HPSG that relies heavily on the notion of a model of a description in a signature. The present paper modifies an elegant method due to Blackburn and Spaan (1993) in order to show that Π^0_1-completeness is the least upper bound (but not a lower bound) on the complexity of deciding if an arbitrary description in a finite or computable signature has a nontrivial model.

2. Complexity Theory Prerequisites

Complexity theory has a rich and extensive literature in which occasionally very similar terms mean very different things, and more frequently very different terms mean very similar things. To avoid confusion, we define here the few complexity classes and give the small number of standard or easily derived complexity theory results the present paper needs. For each $X \subseteq \mathbb{N}$,

> X is **decidable** (or **recursive**)
> iff for some total recursive function ρ from $\mathbb{N}$ to $\mathbb{N}$, for each $n \in \mathbb{N}$,
>
> $n \in X$ iff $\rho(n) = 0$,
>
> X is $\boldsymbol{\Sigma^0_1}$ (or **recursively enumerable**)
> iff for some partial recursive function ρ from $\mathbb{N}$ to $\mathbb{N}$, for each $n \in \mathbb{N}$,
>
> $n \in X$ iff $\rho(n)$ is defined, and
>
> X is $\boldsymbol{\Pi^1_0}$ (or **re-recursively enumerable**)
> iff for some partial recursive function ρ from $\mathbb{N}$ to $\mathbb{N}$, for each $n \in \mathbb{N}$,
>
> $n \in X$ iff $\rho(n)$ is not defined.

PROPOSITION 1. *For each* $X \subseteq \mathbb{N}$,

> X *is* Π^0_1 *iff* $\mathbb{N} \setminus X$ *is* Σ^0_1.

PROPOSITION 2. *For each* $X \subseteq \mathbb{N}$,

> X *is decidable iff* X *is* Σ^0_1 *and* Π^0_1.

Proof. See Enderton (1977: theorem 6.2). □

PROPOSITION 3. *For some* $X \subseteq \mathbb{N}$,

X *is* Σ^0_1 *and not decidable.*

Proof. See Enderton (1977: theorem 4.2 and p. 543). □

PROPOSITION 4. *For some* $X \subseteq \mathbb{N}$,

X *is* Π^0_1 *and not decidable.*

Proof. For some $X \subseteq \mathbb{N}$,

X is Σ^0_1 and not Π^0_1. by Propositions 3 and 2

Thus, for some $X \subseteq \mathbb{N}$,

$\mathbb{N} \setminus X$ is Π^0_1 and not decidable. by Propositions 1 and 2

□

For each $Y \subseteq \mathbb{N}$, for each $Z \subseteq \mathbb{N}$,

Y **reduces** to Z
iff for some total recursive function ρ from $\mathbb{N}$ to $\mathbb{N}$, for each $n \in \mathbb{N}$,

$n \in Y$ iff $\rho(n) \in Z$.

For each $X \subseteq \mathbb{N}$,

X is **Π^0_1-hard** iff for each $Y \subseteq \mathbb{N}$, if Y is Π^0_1 then Y reduces to X, and
X is **Π^0_1-complete** iff X is Π^0_1 and Π^0_1-hard.

PROPOSITION 5. *For each* $X \subseteq \mathbb{N}$,

if X *is* Π^0_1*-hard then* X *is not decidable.*

Proof. For each $X \subseteq \mathbb{N}$,

X is Π^0_1-hard
$\Longrightarrow$ for some $Y \subseteq \mathbb{N}$,
Y reduces to X and Y is not decidable by Proposition 4
$\Longrightarrow$ X is not decidable.

□

As is common in the complexity theory literature, we frequently apply the notions of complexity theory to sets of nonnumbers via some tacit Gödel function. For example, if X is a set and κ is a condition on members of X then we say that $\{x \in X \mid \kappa(x)\}$ is decidable/Σ_1^0/Π_1^0/Π_1^0-hard/Π_1^0-complete to mean that for some implicit effective bijection γ from X to $\mathbb{N}$, $\{\gamma(x) \mid x \in X \text{ and } \kappa(x)\}$ is decidable/Σ_1^0/Π_1^0/Π_1^0-hard/Π_1^0-complete.

3. Speciate Re-Entrant Logic

SRL is a sound, complete (see King, 1989) and decidable (see Kepser, 1994) logic for the HPSG of Pollard and Sag (1994). Here we review only those aspects of the formal language of SRL that are germane to the present paper, and discuss neither the logic of SRL nor the use of SRL to express a HPSG theory (see, instead, King, 1989, 1994; Kepser, 1994; Pollard, 1998).

Underlying a typical interpretable logic is the intuition that an assertion is a finite and syntactically well-formed string of symbols that is either true or false when interpreted. Underlying SRL is the intuition that a description is a finite and syntactically well-formed string of symbols that is either true or false *of an object* when interpreted. For example, the English description “it is black” is true of a soot particle but false of a snow flake when interpreted. To capture these intuitions, SRL provides a class of formal languages, each comprising a signature and a class of interpretations. Each signature provides the nonlogical symbols from which descriptions are syntactically constructed. Each interpretation provides a universe of objects, and assigns each nonlogical symbol a meaning from which is generated a denotation function that assigns each description the collection of objects in the universe of which the description is true. An interpretation satisfies a description iff the description is true of some object in the universe of the interpretation. An interpretation models a description iff the description is true of each object in the universe of the interpretation. Unlike in most interpretable logics, satisfiability and modellability are not synonyms in SRL.

Triple $\langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$ is a **signature** iff
  $\mathbb{S}$ is a set,
  $\mathbb{F}$ is a set, and
  $\mathbb{A}$ is a total function from $\mathbb{S} \times \mathbb{F}$ to $Pow(\mathbb{S})$.

Suppose that $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$ is a signature. We call each member of $\mathbb{S}$ a **species** in Σ, each member of $\mathbb{F}$ a **feature** in Σ, and $\mathbb{A}$ the **appropriateness function** in Σ. For notational facility we henceforth assume that none of the symbols :, $\sim$, $\approx$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, [or] is a species or a feature. For each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$,

Σ is **limited** iff $\mathbb{S}$ is finite,

Σ is **computable** iff $\mathbb{S}$ is finite, $\mathbb{F}$ is countable and $\mathbb{A}$ is recursive, and
Σ is **finite** iff $\mathbb{S}$ is finite and $\mathbb{F}$ is finite.

Notice that each finite signature is computable, and each computable signature is limited. For each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$,

triple $\langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ is an **interpretation** of Σ
iff $\mathcal{U}$ is a set,
  $\mathcal{S}$ is a total function from $\mathcal{U}$ to $\mathbb{S}$,
  $\mathcal{F}$ is a total function from $\mathbb{F}$ to the set of partial functions from $\mathcal{U}$ to $\mathcal{U}$, and
  for each $\varphi \in \mathbb{F}$, for each $o \in \mathcal{U}$,
    $\mathcal{F}(\varphi)(o)$ is defined iff $\mathbb{A}\langle \mathcal{S}(o), \varphi \rangle \neq \emptyset$, and
    if $\mathcal{F}(\varphi)(o)$ is defined then $\mathcal{S}(\mathcal{F}(\varphi)(o)) \in \mathbb{A}\langle \mathcal{S}(o), \varphi \rangle$.

Suppose that $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$ is a signature and $\mathcal{I} = \langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ is an interpretation of Σ. We call $\mathcal{U}$ the **universe** in $\mathcal{I}$, each member of $\mathcal{U}$ an **object** in $\mathcal{I}$, $\mathcal{S}$ the **species assignment function** in $\mathcal{I}$, and $\mathcal{F}$ the **feature denotation function** in $\mathcal{I}$. Each object in $\mathcal{I}$ is of one and only one species, and $\mathcal{S}$ assigns each object in $\mathcal{I}$ its unique species. Each species denotes the set of objects of that species: for each $\sigma \in \mathbb{S}$, σ denotes $\{o \in \mathcal{U} \mid \mathcal{S}(o) = \sigma\}$. Each feature denotes a partial function from $\mathcal{U}$ to $\mathcal{U}$, and $\mathcal{F}$ assigns each feature the partial function it denotes: for each $\varphi \in \mathbb{F}$, φ denotes $\mathcal{F}(\varphi)$. The appropriateness function encodes – and the last line in the definition of an interpretation enforces – a relationship between the denotations of species and features: for each $\sigma \in \mathbb{S}$, for each $\varphi \in \mathbb{F}$, if $\mathbb{A}\langle \sigma, \varphi \rangle = \emptyset$ then the denotation of φ acts upon no object in the denotation of σ, and if $\mathbb{A}\langle \sigma, \varphi \rangle \neq \emptyset$ then the denotation of φ acts upon each object in the denotation of σ to yield an object in the denotation of some species in $\mathbb{A}\langle \sigma, \varphi \rangle$. For each signature Σ, for each interpretation $\mathcal{I} = \langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ of Σ,

$\mathcal{I}$ is **trivial** iff $\mathcal{U} = \emptyset$.

For each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$, $\mathbb{T}_\Sigma$ and $\mathbb{D}_\Sigma$ are the smallest sets such that

$: \in \mathbb{T}_\Sigma$,
for each $\tau \in \mathbb{T}_\Sigma$, for each $\varphi \in \mathbb{F}$, $\tau\varphi \in \mathbb{T}_\Sigma$,
for each $\tau \in \mathbb{T}_\Sigma$, for each $\sigma \in \mathbb{S}$, $\tau \sim \sigma \in \mathbb{D}_\Sigma$,
for each $\tau_1 \in \mathbb{T}_\Sigma$, for each $\tau_2 \in \mathbb{T}_\Sigma$, $\tau_1 \approx \tau_2 \in \mathbb{D}_\Sigma$,
for each $\delta \in \mathbb{D}_\Sigma$, $\neg\delta \in \mathbb{D}_\Sigma$,
for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$, $[\delta_1 \wedge \delta_2] \in \mathbb{D}_\Sigma$,

for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$, $[\delta_1 \vee \delta_2] \in \mathbb{D}_\Sigma$, and

for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$, $[\delta_1 \rightarrow \delta_2] \in \mathbb{D}_\Sigma$.

Suppose that Σ is a signature. We call each member of $\mathbb{T}_\Sigma$ a **term** in Σ, and each member of $\mathbb{D}_\Sigma$ a **description** in Σ. For each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$, for each interpretation $\mathcal{I} = \langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ of Σ,

$\mathcal{T}_\mathcal{I}$ is the total function from $\mathbb{T}_\Sigma$ to the set of partial functions from $\mathcal{U}$ to $\mathcal{U}$ such that

for each $o \in \mathcal{U}$,

$\mathcal{T}_\mathcal{I}(:)(o)$ is defined and $\mathcal{T}_\mathcal{I}(:)(o) = o$, and

for each $\tau \in \mathbb{T}_\Sigma$, for each $\varphi \in \mathbb{F}$, for each $o \in \mathcal{U}$,

$\mathcal{T}_\mathcal{I}(\tau\varphi)(o)$ is defined iff $\mathcal{T}_\mathcal{I}(\tau)(o)$ and $\mathcal{F}(\varphi)(\mathcal{T}_\mathcal{I}(\tau)(o))$ are defined, and

if $\mathcal{T}_\mathcal{I}(\tau\varphi)(o)$ is defined then $\mathcal{T}_\mathcal{I}(\tau\varphi)(o) = \mathcal{F}(\varphi)(\mathcal{T}_\mathcal{I}(\tau)(o))$, and

$\mathcal{D}_\mathcal{I}$ is the total function from $\mathbb{D}_\Sigma$ to *Pow*$(\mathcal{U})$ such that

for each $\tau \in \mathbb{T}_\Sigma$, for each $\sigma \in \mathbb{S}$,

$$\mathcal{D}_\mathcal{I}(\tau \sim \sigma) = \left\{ o \in \mathcal{U} \;\middle|\; \begin{array}{l} \mathcal{T}_\mathcal{I}(\tau)(o) \text{ is defined, and} \\ \mathcal{S}(\mathcal{T}_\mathcal{I}(\tau)(o)) = \sigma \end{array} \right\},$$

for each $\tau_1 \in \mathbb{T}_\Sigma$, for each $\tau_2 \in \mathbb{T}_\Sigma$,

$$\mathcal{D}_\mathcal{I}(\tau_1 \approx \tau_2) = \left\{ o \in \mathcal{U} \;\middle|\; \begin{array}{l} \mathcal{T}_\mathcal{I}(\tau_1)(o) \text{ is defined,} \\ \mathcal{T}_\mathcal{I}(\tau_2)(o) \text{ is defined, and} \\ \mathcal{T}_\mathcal{I}(\tau_1)(o) = \mathcal{T}_\mathcal{I}(\tau_2)(o) \end{array} \right\},$$

for each $\delta \in \mathbb{D}_\Sigma$, $\mathcal{D}_\mathcal{I}(\neg\delta) = \mathcal{U} \setminus \mathcal{D}_\mathcal{I}(\delta)$,

for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$, $\mathcal{D}_\mathcal{I}([\delta_1 \wedge \delta_2]) = \mathcal{D}_\mathcal{I}(\delta_1) \cap \mathcal{D}_\mathcal{I}(\delta_2)$,

for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$, $\mathcal{D}_\mathcal{I}([\delta_1 \vee \delta_2]) = \mathcal{D}_\mathcal{I}(\delta_1) \cup \mathcal{D}_\mathcal{I}(\delta_2)$, and

for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$, $\mathcal{D}_\mathcal{I}([\delta_1 \rightarrow \delta_2]) = (\mathcal{U} \setminus \mathcal{D}_\mathcal{I}(\delta_1)) \cup \mathcal{D}_\mathcal{I}(\delta_2)$.

Suppose that Σ is a signature and $\mathcal{I}$ is an interpretation of Σ. We call $\mathcal{T}_\mathcal{I}$ the **term denotation function** in $\mathcal{I}$, and $\mathcal{D}_\mathcal{I}$ the **description denotation function** in $\mathcal{I}$. A term is the symbol : followed by a finite string of features, and denotes the functional composition of the denotations of its constituent features. Description $\tau \sim \sigma$ is true of object o iff term τ denotes a function defined on o to yield an object in the denotation of species σ. Description $\tau_1 \approx \tau_2$ is true of object o iff terms τ_1 and τ_2 denote functions defined on o to yield one and the same object. Description $\neg\delta$ is true of object o iff δ is false of o. Description $[\delta_1 \wedge \delta_2]$ is true of object o iff δ_1 is true of o and δ_2 is true of o. Description $[\delta_1 \vee \delta_2]$ is true of object o iff δ_1 is true of o or δ_2 is true of o (or both). Description $[\delta_1 \rightarrow \delta_2]$ is true of object o iff δ_2 is true of o whenever δ_1 is true of o. Each description denotes the set of objects of which it is true.

We introduce a number of convenient notational conventions for signifying terms and descriptions. Firstly, for each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$, for each $\varphi \in \mathbb{F}$,

φ^0 signifies the empty string and for each $n \in \mathbb{N}$, φ^{n+1} signifies $\varphi^n\varphi$.

Secondly, for each signature Σ,

for each $\delta_0 \in \mathbb{D}_\Sigma$, $[\![\delta_0]\!]$ signifies δ_0, and
for each $n \in \mathbb{N}$, for each $\delta_0 \in \mathbb{D}_\Sigma$, ..., for each $\delta_n \in \mathbb{D}_\Sigma$, for each $\delta_{n+1} \in \mathbb{D}_\Sigma$,
$[\![\delta_0 \wedge \ldots \wedge \delta_n \wedge \delta_{n+1}]\!]$ signifies $[[\![\delta_0 \wedge \ldots \wedge \delta_n]\!] \wedge \delta_{n+1}]$, and
$[\![\delta_0 \vee \ldots \vee \delta_n \vee \delta_{n+1}]\!]$ signifies $[[\![\delta_0 \vee \ldots \vee \delta_n]\!] \vee \delta_{n+1}]$.

Thirdly, for each signature Σ,

$\bigwedge \emptyset$ signifies $: \approx :$,
$\bigvee \emptyset$ signifies $\neg : \approx :$, and
for each finite and nonempty $\theta \subseteq \mathbb{D}_\Sigma$,
$\bigwedge \theta$ signifies $[\![\delta_0 \wedge \ldots \wedge \delta_n]\!]$, and
$\bigvee \theta$ signifies $[\![\delta_0 \vee \ldots \vee \delta_n]\!]$,

where $\langle \delta_0, \ldots, \delta_n \rangle$ is some fixed well-ordering of θ.

For each signature Σ, for each $\delta \in \mathbb{D}_\Sigma$, for each interpretation $\mathcal{I} = \langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ of Σ,

$\mathcal{I}$ **satisfies** δ in Σ iff $\mathcal{D}_\mathcal{I}(\delta) \neq \emptyset$, and
$\mathcal{I}$ **models** δ in Σ iff $\mathcal{D}_\mathcal{I}(\delta) = \mathcal{U}$.

Notice that for each signature Σ, for each $\delta \in \mathbb{D}_\Sigma$, for each trivial interpretation $\mathcal{I}$ of Σ, $\mathcal{I}$ models δ in Σ. For each signature Σ, for each $\delta \in \mathbb{D}_\Sigma$,

δ is **satisfiable** in Σ iff for some interpretation $\mathcal{I}$ of Σ, $\mathcal{I}$ satisfies δ in Σ, and
δ is **modellable** in Σ iff for some nontrivial interpretation $\mathcal{I}$ of Σ, $\mathcal{I}$ models δ in Σ.

SRL differs from most interpretable logics in that satisfiability and modellability are not synonyms: though every modellable description is satisfiable, not every satisfiable description is modellable. For example, suppose that

$$\mathbb{S} = \{\mathit{wife}, \mathit{husband}\},$$

$$\mathbb{F} = \{\textsf{spouse}\},$$

$$\mathbb{A} \text{ is the total function from } \mathbb{S} \times \mathbb{F} \text{ to } \mathit{Pow}(\mathbb{S}) \text{ such that}$$

$$\mathbb{A}\langle \mathit{wife}, \textsf{spouse}\rangle = \{\mathit{husband}\} \text{ and } \mathbb{A}\langle \mathit{husband}, \textsf{spouse}\rangle = \{\mathit{wife}\},$$

$$\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A}\rangle, \text{ and}$$

$$\delta = [: \sim \mathit{wife} \ \wedge \ :\textsf{spouse} \sim \mathit{husband}].$$

Then Σ is a signature and $\delta \in \mathbb{D}_\Sigma$. Clearly, δ is satisfiable: an interpretation comprising one wife and one husband married to each other satisfies δ. Equally clearly, δ is not modellable: a nontrivial interpretation that models δ can only comprise wives, yet each wife must have a husband as the value of their spouse feature.

Satisfiers are linguistically significant. For example, consider a test of the English language grammaticality of the sentence “I think.” The test comprises the phonological string /aɪ θɪŋk/, and is passed iff some sentence in the English language has this string as its phonological component. Under the admittedly naïve Pollard and Sag (1994) treatment of phonology, the test can be rendered as the SRL description

$$\llbracket : \sim \mathit{phrase} \ \wedge \ :\textsf{synsem local category subcat} \sim \mathit{empty\text{-}list} \ \wedge$$

$$:\textsf{phonology first} \sim \text{aɪ} \ \wedge \ :\textsf{phonology rest first} \sim \text{θɪŋk} \ \wedge$$

$$:\textsf{phonology rest rest} \sim \mathit{empty\text{-}list}\rrbracket,$$

and is passed iff the English language, construed as a SRL interpretation, satisfies the SRL description. The decidability of satisfiability is thus of some linguistic utility, and Kepser (1994) shows that satisfiability is decidable in each computable signature: for each computable signature Σ, $\{\delta \in \mathbb{D}_\Sigma \mid \delta \text{ is satisfiable in } \Sigma\}$ is decidable.

Models are also linguistically significant. For example, the Pollard and Sag (1994) theory of English comprises a set of principles, and is true iff each principle in the theory describes each object in the English language. Each principle in the theory can be rendered as a SRL description: the head feature principle is, simplifying slightly, rendered as

$$[[: \sim \mathit{phrase} \ \wedge \ :\textsf{daughters} \sim \mathit{headed\text{-}structure}]$$

$$\rightarrow \ :\textsf{synsem local category head} \approx$$

$$:\textsf{daughters head-dtr synsem local category head}].$$

The theory can then be rendered as the conjunction of the SRL renderings of each of its constituent principles, and is true iff the English language, again construed as a SRL interpretation, models the conjunction.[⋆] The decidability of modellability is thus also of some linguistic utility. Unfortunately, the next section shows that modellability is not decidable in each signature, not even in each finite signature: for some finite signature Σ, $\{\delta \in \mathbb{D}_\Sigma \mid \delta \text{ is modellable in } \Sigma\}$ is Π^0_1-hard, and hence not decidable.

4. The Complexity of Modellability

Blackburn and Spaan (1993) elegantly prove a series of complexity results for satisfiability in a family of propositional modal logics, including a logic $L^{KR\Box}$ named for Kasper–Rounds logic (see Kasper and Rounds, 1986, 1990) augmented with master modality $\Box$. $L^{KR\Box}$ satisfiability is very similar to SRL modellability, but differences between $L^{KR\Box}$ and SRL unfortunately mean that the Blackburn and Spaan (1993) theorems concerning the complexity of $L^{KR\Box}$ satisfiability do not apply directly to SRL modellability. However, in this section, we modify the proofs of Blackburn and Spaan (1993: theorems 4.1, 4.2 and 4.6) and furnish new proofs of our own in order to show that

- SRL modellability in each computable signature is Π^0_1,
- SRL modellability in some finite signature is Π^0_1-hard, and
- SRL modellability in some finite signature is decidable.

4.1. MODELLABILITY IN EACH COMPUTABLE SIGNATURE IS Π^0_1

Blackburn and Spaan (1993) show that $L^{KR\Box}$ satisfiability in certain $L^{KR\Box}$ signatures is Π^0_1 by using (van Benthem, 1984) to correlate $L^{KR\Box}$ nonsatisfiability in certain $L^{KR\Box}$ signatures and FOL (first-order predicate logic) inference in certain FOL signatures. We similarly show that SRL modellability in computable SRL signatures is Π^0_1 by using (Aldag, 1997) to correlate SRL nonmodellability in limited SRL signatures and FOL inference in certain FOL signatures. We assume throughout this section that the reader is familiar with the rudiments of FOL.

Aldag (1997) first translates each limited SRL signature into a FOL signature and an associated theory. If $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$ is a SRL signature then Aldag (1997) considers $\overline{\Sigma} = \langle \mathbb{S}, \mathbb{F} \rangle$ a FOL signature in which each member of $\mathbb{S}$ is a unary relation symbol and each member of $\mathbb{F}$ is a binary relation symbol. We assume henceforth that $\mathbb{V}$ is a fixed countably infinite set, and $\langle x, y, z, y_0, z_0, y_1, z_1, y_2, z_2, \ldots \rangle$

[⋆] In fact, the theory also comprises a lexicon, some lexical rules and some linear precedence rules. However, all of these are expressible within SRL. For example, King and Simov (1998) give an effective algorithm for automatically deducing from a finite SRL theory a classificatory system suitable for the construction of a HPSG lexicon, Meurers and Minnen (1997) present an approach to lexical rules that treats each lexical rule as a SRL description, and Richter and Sailer (1995) use SRL to provide a broad yet rigorous HPSG account of word order in the German Mittelfeld.

is a fixed well-ordering of $\mathbb{V}$. For each SRL signature Σ, $\mathbb{W}_{\overline{\Sigma}}$ is the set of FOL formulae in FOL signature $\overline{\Sigma}$ with equality $\approx$ and variables $\mathbb{V}$. However, suppose that $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$ is a SRL signature. A FOL interpretation of $\overline{\Sigma}$ need not be a SRL interpretation of Σ, since nothing enforces on each FOL interpretation of $\overline{\Sigma}$ the relationship between the interpretations of members of $\mathbb{S}$ and $\mathbb{F}$ that $\mathbb{A}$ enforces on each SRL interpretation of Σ. Instead, for each limited SRL signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$, Aldag (1997) captures $\mathbb{A}$ as a set of FOL sentences in $\overline{\Sigma}$. For each limited SRL signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$,

$$
\begin{aligned}
\mathtt{Sign}(\Sigma) = {} & \left\{ \forall x \bigvee \{\sigma[x] \mid \sigma \in \mathbb{S}\} \right\} \\
& \cup \left\{ \forall x \, \neg[\sigma_1[x] \wedge \sigma_2[x]] \;\middle|\; \begin{array}{l} \sigma_1 \in \mathbb{S}, \\ \sigma_2 \in \mathbb{S}, \text{ and} \\ \sigma_1 \neq \sigma_2 \end{array} \right\} \\
& \cup \left\{ \forall x \, \forall y \, \forall z \, [[\varphi[x, y] \wedge \varphi[x, z]] \rightarrow y \approx z] \;\middle|\; \varphi \in \mathbb{F} \right\} \\
& \cup \left\{ \forall x \, [\sigma[x] \rightarrow \exists y \, [\varphi[x, y] \wedge \bigvee \{\sigma'[y] \mid \sigma' \in \mathbb{A}\langle \sigma, \varphi \rangle\}]] \;\middle|\; \begin{array}{l} \sigma \in \mathbb{S}, \\ \varphi \in \mathbb{F}, \text{ and} \\ \mathbb{A}\langle \sigma, \varphi \rangle \neq \emptyset \end{array} \right\} \\
& \cup \left\{ \forall x \, [\sigma[x] \rightarrow \neg \exists y \, \varphi[x, y]] \;\middle|\; \begin{array}{l} \sigma \in \mathbb{S}, \\ \varphi \in \mathbb{F}, \text{ and} \\ \mathbb{A}\langle \sigma, \varphi \rangle = \emptyset \end{array} \right\}
\end{aligned}
$$

For each Y, for each Z, for each $R \subseteq Y \times Z$, $\downarrow R$ is the total function from Z to $Pow(Y)$ such that for each $z \in Z$,

$$\downarrow R(z) = \{y \in Y \mid \langle y, z \rangle \in R\}.$$

PROPOSITION 6. *For each limited SRL signature* Σ, *for each triple* $\langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$,

$\langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ *is a nontrivial SRL interpretation of* Σ
iff $\downarrow \mathcal{S}$ *is defined,*
$\langle \mathcal{U}, \downarrow \mathcal{S}, \mathcal{F} \rangle$ *is a FOL interpretation of* $\overline{\Sigma}$, *and*
$\langle \mathcal{U}, \downarrow \mathcal{S}, \mathcal{F} \rangle$ *FOL satisfies* $\mathtt{Sign}(\Sigma)$ *in* $\overline{\Sigma}$.

Aldag (1997) next translates SRL descriptions into FOL formulae with unique free variable x under the intuition that a description δ is true of an object o iff the translation of δ is true under a valuation that assigns o to x. For each SRL signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$, $\mathtt{Desc}_{\Sigma}$ is the total function from $\mathbb{D}_{\Sigma}$ to $\mathbb{W}_{\overline{\Sigma}}$ such that for each $n \in \mathbb{N}$, for each $\varphi_1 \in \mathbb{F}$, ..., for each $\varphi_n \in \mathbb{F}$, for each $\sigma \in \mathbb{S}$,

$\mathtt{Desc}_\Sigma(:\varphi_1 \ldots \varphi_n \sim \sigma)$

$= \exists y_0 \ \ldots \exists y_n \ [\![x \approx y_0 \wedge \varphi_1[y_0, y_1] \wedge \ldots \wedge \varphi_n[y_{n-1}, y_n] \wedge \sigma[y_n]]\!]$,

for each $m \in \mathbb{N}$, for each $n \in \mathbb{N}$, for each $\varphi_1 \in \mathbb{F}$, ..., for each $\varphi_m \in \mathbb{F}$,

for each $\varphi'_1 \in \mathbb{F}$, ..., for each $\varphi'_n \in \mathbb{F}$,

$\mathtt{Desc}_\Sigma(:\varphi_1 \ldots \varphi_m \approx :\varphi'_1 \ldots \varphi'_n)$

$$= \exists y_0 \ \ldots \exists y_m \ \exists z_0 \ \ldots \exists z_n \ [\![x \approx y_0 \wedge \varphi_1[y_0, y_1] \wedge \ldots \wedge \varphi_m[y_{m-1}, y_m] \wedge x \approx z_0 \wedge \varphi'_1[z_0, z_1] \wedge \ldots \wedge \varphi'_n[z_{n-1}, z_n] \wedge y_m \approx z_n]\!],$$

for each $\delta \in \mathbb{D}_\Sigma$, $\mathtt{Desc}_\Sigma(\neg\delta) = \neg\mathtt{Desc}_\Sigma(\delta)$,

for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$,

$\mathtt{Desc}_\Sigma([\delta_1 \wedge \delta_2]) = [\mathtt{Desc}_\Sigma(\delta_1) \wedge \mathtt{Desc}_\Sigma(\delta_2)]$,

for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$,

$\mathtt{Desc}_\Sigma([\delta_1 \vee \delta_2]) = [\mathtt{Desc}_\Sigma(\delta_1) \vee \mathtt{Desc}_\Sigma(\delta_2)]$, and

for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$,

$\mathtt{Desc}_\Sigma([\delta_1 \rightarrow \delta_2]) = [\mathtt{Desc}_\Sigma(\delta_1) \rightarrow \mathtt{Desc}_\Sigma(\delta_2)]$.

PROPOSITION 7. *For each limited SRL signature* Σ, *for each triple* $\langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$, *for each* $\delta \in \mathbb{D}_\Sigma$, *for each* $o \in \mathcal{U}$,

$\langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ *is a nontrivial SRL interpretation of* Σ, *and*
$o \in \mathcal{D}_{\mathcal{I}}(\delta)$
iff $\downarrow\mathcal{S}$ *is defined,*
$\langle \mathcal{U}, \downarrow\mathcal{S}, \mathcal{F} \rangle$ *is a FOL interpretation of* $\overline{\Sigma}$,
$\langle \mathcal{U}, \downarrow\mathcal{S}, \mathcal{F} \rangle$ *FOL satisfies* $\mathtt{Sign}(\Sigma)$ *in* $\overline{\Sigma}$, *and*
for each total function υ *from* $\mathbb{V}$ *to* $\mathcal{U}$,
if $\upsilon(x) = o$ *then* $\langle \mathcal{U}, \downarrow\mathcal{S}, \mathcal{F} \rangle$ *FOL satisfies* $\mathtt{Desc}_\Sigma(\delta)$ *under* υ *in* $\overline{\Sigma}$.

Proof. By Proposition 6 and arithmetic induction on the complexity of δ. □

Aldag (1997) uses the foregoing translations to correlate – among other things – SRL modellability in a limited SRL signature and FOL satisfiability in the corresponding FOL signature.

PROPOSITION 8. *For each limited SRL signature* Σ, *for each* $\delta \in \mathbb{D}_\Sigma$,

δ *is SRL modellable in* Σ *iff* $\mathtt{Sign}(\Sigma) \cup \{\forall x\, \mathtt{Desc}_\Sigma(\delta)\}$ *is FOL satisfiable in* $\overline{\Sigma}$.

Proof. Firstly, for each limited SRL signature Σ, for each $\delta \in \mathbb{D}_\Sigma$,

if δ is SRL modellable in Σ
then $\texttt{Sign}(\Sigma) \cup \{\forall x\ \texttt{Desc}_\Sigma(\delta)\}$ is FOL satisfiable in $\overline{\Sigma}$. by Proposition 7

Secondly, suppose for each set Y, for each set Z, for each total function F from Z to $Pow(Y)$,

$$\uparrow F = \{\langle y, z\rangle \in Y \times Z \mid y \in F(z)\}.$$

Thus, for each set Y, for each set Z, for each total function F from Z to $Pow(Y)$,

$$\downarrow\uparrow F = F.$$

Thus, for each limited SRL signature Σ, for each $\delta \in \mathbb{D}_\Sigma$,

$\texttt{Sign}(\Sigma) \cup \{\forall x\ \texttt{Desc}_\Sigma(\delta)\}$ is FOL satisfiable in $\overline{\Sigma}$
$\Longrightarrow$ for some triple $\langle \mathcal{U}, \mathcal{S}, \mathcal{F}\rangle$,
$\langle \mathcal{U}, \downarrow\uparrow\mathcal{S}, \mathcal{F}\rangle$ is a FOL interpretation of $\overline{\Sigma}$, and
$\langle \mathcal{U}, \downarrow\uparrow\mathcal{S}, \mathcal{F}\rangle$ FOL satisfies $\texttt{Sign}(\Sigma) \cup \{\forall x\ \texttt{Desc}_\Sigma(\delta)\}$ in $\overline{\Sigma}$
$\Longrightarrow$ for some triple $\langle \mathcal{U}, \mathcal{S}, \mathcal{F}\rangle$,
$\langle \mathcal{U}, \uparrow\mathcal{S}, \mathcal{F}\rangle$ is a nontrivial SRL interpretation of Σ, and
$\langle \mathcal{U}, \uparrow\mathcal{S}, \mathcal{F}\rangle$ SRL models δ in Σ by Proposition 7
$\Longrightarrow$ δ is SRL modellable in Σ.

□

We can, therefore, correlate SRL nonmodellability in a limited SRL signature and FOL inference in the corresponding FOL signature.

PROPOSITION 9. *For each limited SRL signature* Σ, *for each* $\delta \in \mathbb{D}_\Sigma$,

δ *is not SRL modellable in* Σ *iff* $\texttt{Sign}(\Sigma)$ *FOL infers* $\neg\forall x\ \texttt{Desc}_\Sigma(\delta)$ *in* $\overline{\Sigma}$.

Proof. For each limited SRL signature Σ, for each $\delta \in \mathbb{D}_\Sigma$,

δ is not SRL modellable in Σ
$\Longleftrightarrow$ $\texttt{Sign}(\Sigma) \cup \{\forall x\ \texttt{Desc}_\Sigma(\delta)\}$ is not FOL satisfiable in $\overline{\Sigma}$ by Proposition 8
$\Longleftrightarrow$ $\texttt{Sign}(\Sigma)$ FOL entails $\neg\forall x\ \texttt{Desc}_\Sigma(\delta)$ in $\overline{\Sigma}$
$\Longleftrightarrow$ $\texttt{Sign}(\Sigma)$ FOL infers $\neg\forall x\ \texttt{Desc}_\Sigma(\delta)$ in $\overline{\Sigma}$.

□

We can, therefore, deduce that modellability in each computable signature is Π^0_1.

THEOREM 10. *For each computable SRL signature* Σ,

$$\{\delta \in \mathbb{D}_\Sigma \mid \delta \textit{ is modellable in } \Sigma\} \textit{ is } \Pi^0_1.$$

Proof. For each SRL signature Σ,

$$\begin{array}{ll}
\Sigma \text{ is computable} & \\
\Longrightarrow \mathbb{W}_{\overline{\Sigma}} \text{ is countable and } \{\omega \in \mathbb{W}_{\overline{\Sigma}} \mid \omega \in \mathtt{Sign}(\Sigma)\} \text{ is decidable} & \\
\Longrightarrow \{\omega \in \mathbb{W}_{\overline{\Sigma}} \mid \omega \text{ is sentential and } \mathtt{Sign}(\Sigma) \text{ FOL infers } \omega \text{ in } \overline{\Sigma}\} \text{ is } \Sigma^0_1 & \\
\Longrightarrow \{\delta \in \mathbb{D}_\Sigma \mid \mathtt{Sign}(\Sigma) \text{ FOL infers } \neg\forall x\, \mathtt{Desc}_\Sigma(\delta) \text{ in } \overline{\Sigma}\} \text{ is } \Sigma^0_1 & \\
\Longrightarrow \{\delta \in \mathbb{D}_\Sigma \mid \delta \text{ is not SRL modellable in } \Sigma\} \text{ is } \Sigma^0_1 & \text{by Proposition 9} \\
\Longrightarrow \{\delta \in \mathbb{D}_\Sigma \mid \delta \text{ is SRL modellable in } \Sigma\} \text{ is } \Pi^0_1. & \text{by Proposition 1}
\end{array}$$

□

4.2. Modellability in Some Finite Signature Is Π^0_1-Hard

Blackburn and Spaan (1993) base a very elegant proof that satisfiability in certain $L^{KR\Box}$ signatures is Π^0_1-hard on the Π^0_1-completeness of a tiling problem. Informally, a domino is a square with a fixed orientation and each edge bearing a natural number. Two dominoes are alike iff their right edges bear the same number, their left edges bear the same number, their top edges bear the same number, and their bottom edges bear the same number. Alikeness is an equivalence relation on dominoes, and a domino type is an alikeness class of dominoes. A set of domino types tiles $\mathbb{N}^2$ iff dominoes of these types can cover a quadrant of the infinite chess board in such a way that the adjacent edges of neighboring dominoes bear identical numbers. It is known that the problem of deciding whether an arbitrary finite and nonempty set of domino types can tile $\mathbb{N}^2$ is Π^0_1-complete. Formally,

$$\mathfrak{T} = \mathbb{N}^4,$$

$$\mathfrak{P} = \left\{ \langle \mathfrak{t}_0, \ldots, \mathfrak{t}_n \rangle \;\middle|\; \begin{array}{l} n \in \mathbb{N}, \{\mathfrak{t}_0, \ldots, \mathfrak{t}_n\} \subseteq \mathfrak{T}, \text{ and} \\ \text{for each } i \in \{0, \ldots, n\}, \text{for each } j \in \{0, \ldots, n\}, \\ \quad \text{if } i \neq j \text{ then } \mathfrak{t}_i \neq \mathfrak{t}_j \end{array} \right\}, \text{ and}$$

$$\mathfrak{A} = \left\{ \mathfrak{a} \;\middle|\; \mathfrak{a} \text{ is a total function from } \mathbb{N}^2 \text{ to } \mathbb{N} \right\}.$$

We call each member of $\mathfrak{T}$ a **type**, each member of $\mathfrak{P}$ a **problem**, and each member of $\mathfrak{A}$ an **assignment**. For each $\langle r, l, t, b\rangle \in \mathfrak{T}$,

$$\mathfrak{Right}\langle r, l, t, b\rangle = r,$$

$\mathfrak{Left}\langle r, l, t, b\rangle = l$,
$\mathfrak{Top}\langle r, l, t, b\rangle = t$, and
$\mathfrak{Bottom}\langle r, l, t, b\rangle = b$.

For each $n \in \mathbb{N}$, for each $\mathfrak{p} = \langle \mathfrak{t}_0, \ldots, \mathfrak{t}_n \rangle \in \mathfrak{P}$, for each $\mathfrak{a} \in \mathfrak{A}$,

$\mathfrak{a}$ **solves** $\mathfrak{p}$
iff for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$,
$\mathfrak{a}\langle i, j\rangle \in \{0, \ldots, n\}$,
$\mathfrak{Right}(\mathfrak{t}_{\mathfrak{a}\langle i,j\rangle}) = \mathfrak{Left}(\mathfrak{t}_{\mathfrak{a}\langle i+1,j\rangle})$, and
$\mathfrak{Top}(\mathfrak{t}_{\mathfrak{a}\langle i,j\rangle}) = \mathfrak{Bottom}(\mathfrak{t}_{\mathfrak{a}\langle i,j+1\rangle})$.

For each $\mathfrak{p} \in \mathfrak{P}$,

$\mathfrak{p}$ is **soluble** iff for some $\mathfrak{a} \in \mathfrak{A}$, $\mathfrak{a}$ solves $\mathfrak{p}$.

Berger (1966) and Robinson (1971) prove the following proposition.

PROPOSITION 11. $\{\mathfrak{p} \in \mathfrak{P} \mid \mathfrak{p}$ *is soluble*$\}$ *is* Π^0_1*-complete.*

We modify the Blackburn and Spaan (1993) proof that $L^{KR\Box}$ satisfiability in certain $L^{KR\Box}$ signatures is Π^0_1-hard in order to yield a similar proof that SRL modellability in some finite SRL signature is Π^0_1-hard.

We first define a finite signature $\widehat{\Sigma}$ and a total recursive function # from $\mathfrak{P}$ to $\mathbb{D}_{\widehat{\Sigma}}$.

$\widehat{\mathbb{S}} = \{\mathit{node}, \mathit{true}, \mathit{false}\}$,
$\widehat{\mathbb{F}} = \{\mathsf{file}, \mathsf{rank}, \mathsf{level}, \mathsf{next}, \mathsf{earth}\}$,
$\widehat{\mathbb{A}}$ is the total function from $\widehat{\mathbb{S}} \times \widehat{\mathbb{F}}$ to $\mathit{Pow}(\widehat{\mathbb{S}})$ such that
$\widehat{\mathbb{A}}\langle \mathit{node}, \mathsf{file}\rangle = \{\mathit{node}\}$,
$\widehat{\mathbb{A}}\langle \mathit{node}, \mathsf{rank}\rangle = \{\mathit{node}\}$,
$\widehat{\mathbb{A}}\langle \mathit{node}, \mathsf{level}\rangle = \{\mathit{true}, \mathit{false}\}$,
$\widehat{\mathbb{A}}\langle \mathit{node}, \mathsf{next}\rangle = \emptyset$,
$\widehat{\mathbb{A}}\langle \mathit{node}, \mathsf{earth}\rangle = \emptyset$,
$\widehat{\mathbb{A}}\langle \mathit{true}, \mathsf{file}\rangle = \emptyset$,
$\widehat{\mathbb{A}}\langle \mathit{true}, \mathsf{rank}\rangle = \emptyset$,
$\widehat{\mathbb{A}}\langle \mathit{true}, \mathsf{level}\rangle = \emptyset$,
$\widehat{\mathbb{A}}\langle \mathit{true}, \mathsf{next}\rangle = \{\mathit{true}, \mathit{false}\}$,

$$
\begin{aligned}
&\quad \widehat{\mathbb{A}}\langle \mathit{true}, \mathsf{earth}\rangle = \{\mathit{node}\},\\
&\quad \widehat{\mathbb{A}}\langle \mathit{false}, \mathsf{file}\rangle = \emptyset,\\
&\quad \widehat{\mathbb{A}}\langle \mathit{false}, \mathsf{rank}\rangle = \emptyset,\\
&\quad \widehat{\mathbb{A}}\langle \mathit{false}, \mathsf{level}\rangle = \emptyset,\\
&\quad \widehat{\mathbb{A}}\langle \mathit{false}, \mathsf{next}\rangle = \{\mathit{true}, \mathit{false}\}, \text{ and}\\
&\quad \widehat{\mathbb{A}}\langle \mathit{false}, \mathsf{earth}\rangle = \{\mathit{node}\}, \text{ and}\\
&\widehat{\Sigma} = \langle \widehat{\mathbb{S}}, \widehat{\mathbb{F}}, \widehat{\mathbb{A}}\rangle.
\end{aligned}
$$

PROPOSITION 12. *$\widehat{\Sigma}$ is a finite signature.*

$\#_0$, $\#_1$, $\#_2$, $\#_3$, $\#_4$ and $\#$ are the total functions from $\mathfrak{P}$ to $\mathbb{D}_{\widehat{\Sigma}}$ such that for each $n \in \mathbb{N}$, for each $\mathfrak{p} = \langle \mathfrak{t}_0, \ldots, \mathfrak{t}_n\rangle \in \mathfrak{P}$,

$$\#_0(\mathfrak{p}) = {:}\mathsf{file}\ \mathsf{rank} \approx {:}\mathsf{rank}\ \mathsf{file},$$

$$\#_1(\mathfrak{p}) = \bigvee \left\{ {:}\mathsf{level}\ \mathsf{next}^k \sim \mathit{true} \,\middle|\, k \in \{0, \ldots, n\} \right\},$$

$$\#_2(\mathfrak{p}) = \bigwedge \left\{ [{:}\mathsf{level}\ \mathsf{next}^k \sim \mathit{false} \ \vee\ {:}\mathsf{level}\ \mathsf{next}^l \sim \mathit{false}] \,\middle|\, \begin{array}{l} k \in \{0, \ldots, n\},\\ l \in \{0, \ldots, n\}, \text{ and}\\ k \neq l \end{array} \right\},$$

$$\#_3(\mathfrak{p}) = \bigvee \left\{ [{:}\mathsf{level}\ \mathsf{next}^k \sim \mathit{true} \ \wedge\ {:}\mathsf{file}\ \mathsf{level}\ \mathsf{next}^l \sim \mathit{true}] \,\middle|\, \begin{array}{l} k \in \{0, \ldots, n\},\\ l \in \{0, \ldots, n\}, \text{ and}\\ \mathfrak{Right}(\mathfrak{t}_k) = \mathfrak{Left}(\mathfrak{t}_l) \end{array} \right\},$$

$$\#_4(\mathfrak{p}) = \bigvee \left\{ [{:}\mathsf{level}\ \mathsf{next}^k \sim \mathit{true} \ \wedge\ {:}\mathsf{rank}\ \mathsf{level}\ \mathsf{next}^l \sim \mathit{true}] \,\middle|\, \begin{array}{l} k \in \{0, \ldots, n\},\\ l \in \{0, \ldots, n\}, \text{ and}\\ \mathfrak{Top}(\mathfrak{t}_k) = \mathfrak{Bottom}(\mathfrak{t}_l) \end{array} \right\},$$

and

$$\#(\mathfrak{p}) = [{:} \sim \mathit{node} \ \rightarrow\ [\![\#_0(\mathfrak{p}) \ \wedge\ \#_1(\mathfrak{p}) \ \wedge\ \#_2(\mathfrak{p}) \ \wedge\ \#_3(\mathfrak{p}) \ \wedge\ \#_4(\mathfrak{p})]\!]].$$

PROPOSITION 13. *$\#$ is a total recursive function from $\mathfrak{P}$ to $\mathbb{D}_{\widehat{\Sigma}}$.*

We next define for each assignment $\mathfrak{a}$ a nontrivial interpretation $\mathcal{I}_{\mathfrak{a}}$ of $\widehat{\Sigma}$, show that if an assignment $\mathfrak{a}$ solves a problem $\mathfrak{p}$ then $\mathcal{I}_{\mathfrak{a}}$ models $\#(\mathfrak{p})$ in $\widehat{\Sigma}$, and show that a nontrivial interpretation determines a solution of a problem $\mathfrak{p}$ if the interpretation models $\#(\mathfrak{p})$ in $\widehat{\Sigma}$. Thus, a problem $\mathfrak{p}$ is soluble iff $\#(\mathfrak{p})$ is modellable in $\widehat{\Sigma}$. For each $\mathfrak{a} \in \mathfrak{A}$,

$$
\begin{aligned}
&\mathcal{N}_{\mathfrak{a}} = \{\nu_{i,j} \mid i \in \mathbb{N} \text{ and } j \in \mathbb{N}\},\\
&\mathcal{L}_{\mathfrak{a}} = \{\lambda^k_{i,j} \mid i \in \mathbb{N}, j \in \mathbb{N} \text{ and } k \in \mathbb{N}\},
\end{aligned}
$$

$\mathcal{U}_{\mathfrak{a}} = \mathcal{N}_{\mathfrak{a}} \cup \mathcal{L}_{\mathfrak{a}}$,

$\mathcal{S}_{\mathfrak{a}}$ is the total function from $\mathcal{U}_{\mathfrak{a}}$ to $\widehat{\mathbb{S}}$ such that

for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$, $\mathcal{S}_{\mathfrak{a}}(\nu_{i,j}) = \textit{node}$, and

for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$, for each $k \in \mathbb{N}$,

$$\mathcal{S}_{\mathfrak{a}}(\lambda^k_{i,j}) = \begin{cases} \textit{true} & \text{if } \mathfrak{a}\langle i, j\rangle = k \\ \textit{false} & \text{otherwise}, \end{cases}$$

$\mathcal{F}_{\mathfrak{a}}$ is the total function from $\widehat{\mathbb{F}}$ to the set of partial functions from $\mathcal{U}_{\mathfrak{a}}$ to $\mathcal{U}_{\mathfrak{a}}$ such that

$\mathcal{F}_{\mathfrak{a}}(\mathsf{file})$ is the total function from $\mathcal{N}_{\mathfrak{a}}$ to $\mathcal{N}_{\mathfrak{a}}$ such that

for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$, $\mathcal{F}_{\mathfrak{a}}(\mathsf{file})(\nu_{i,j}) = \nu_{i+1,j}$,

$\mathcal{F}_{\mathfrak{a}}(\mathsf{rank})$ is the total function from $\mathcal{N}_{\mathfrak{a}}$ to $\mathcal{N}_{\mathfrak{a}}$ such that

for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$, $\mathcal{F}_{\mathfrak{a}}(\mathsf{rank})(\nu_{i,j}) = \nu_{i,j+1}$,

$\mathcal{F}_{\mathfrak{a}}(\mathsf{level})$ is the total function from $\mathcal{N}_{\mathfrak{a}}$ to $\mathcal{L}_{\mathfrak{a}}$ such that

for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$, $\mathcal{F}_{\mathfrak{a}}(\mathsf{level})(\nu_{i,j}) = \lambda^0_{i,j}$,

$\mathcal{F}_{\mathfrak{a}}(\mathsf{next})$ is the total function from $\mathcal{L}_{\mathfrak{a}}$ to $\mathcal{L}_{\mathfrak{a}}$ such that

for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$, for each $k \in \mathbb{N}$, $\mathcal{F}_{\mathfrak{a}}(\mathsf{next})(\lambda^k_{i,j}) = \lambda^{k+1}_{i,j}$,

and

$\mathcal{F}_{\mathfrak{a}}(\mathsf{earth})$ is the total function from $\mathcal{L}_{\mathfrak{a}}$ to $\mathcal{N}_{\mathfrak{a}}$ such that

for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$, for each $k \in \mathbb{N}$, $\mathcal{F}_{\mathfrak{a}}(\mathsf{earth})(\lambda^k_{i,j}) = \nu_{i,j}$,

and

$\mathcal{I}_{\mathfrak{a}} = \langle \mathcal{U}_{\mathfrak{a}}, \mathcal{S}_{\mathfrak{a}}, \mathcal{F}_{\mathfrak{a}} \rangle$.

LEMMA 14. *For each* $\mathfrak{a} \in \mathfrak{A}$,

$\mathcal{I}_{\mathfrak{a}}$ *is a nontrivial interpretation of* $\widehat{\Sigma}$.

Suppose that $\mathfrak{a} \in \mathfrak{A}$. We call each member of $\mathcal{N}_{\mathfrak{a}}$ a **node** and each member of $\mathcal{L}_{\mathfrak{a}}$ a **level indicator**. The nodes emulate an upper right quadrant of an infinite chessboard, with $\nu_{0,0}$ as the lowest and leftmost node of the quadrant, feature file mapping node $\nu_{i,j}$ to its immediate neighbor $\nu_{i+1,j}$ in the next file rightwards, and feature rank mapping node $\nu_{i,j}$ to its immediate neighbor $\nu_{i,j+1}$ in the next rank upwards. Figure 1 depicts the quadrant emulated by the nodes. Feature level maps node $\nu_{i,j}$ to level indicator $\lambda^0_{i,j}$, the first in an infinite chain $\lambda^0_{i,j}, \lambda^1_{i,j}, \lambda^2_{i,j}, \ldots$ of level indicators, with feature next mapping level indicator $\lambda^k_{i,j}$ to its immediate successor $\lambda^{k+1}_{i,j}$ in the chain, and feature earth mapping level indicator $\lambda^k_{i,j}$ to the source node $\nu_{i,j}$ of the chain. In each chain $\lambda^0_{i,j}, \lambda^1_{i,j}, \lambda^2_{i,j}, \ldots$ of level indicators,

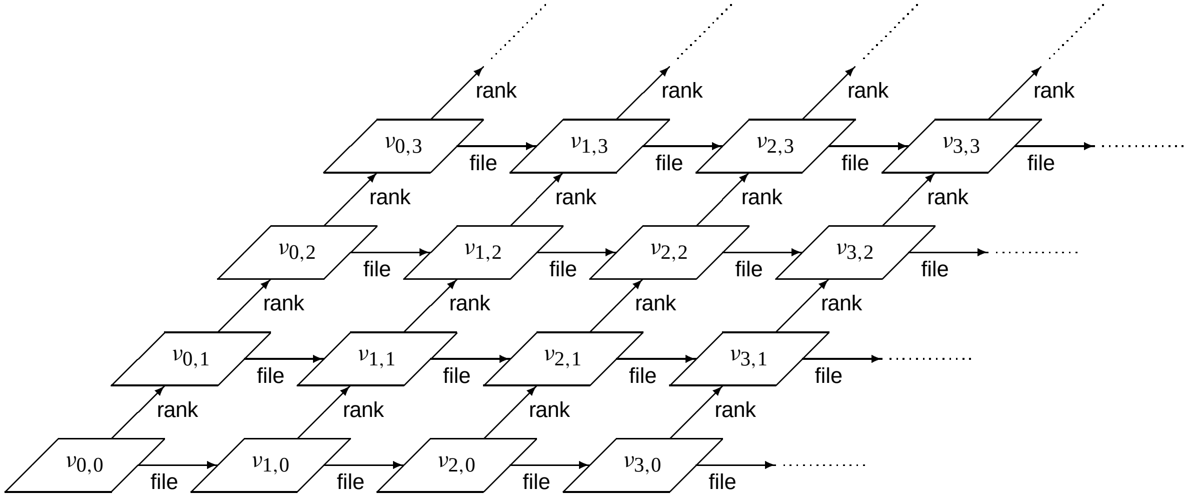

Figure 1. Quadrant emulated by nodes.

level indicator $\lambda_{i,j}^{\mathfrak{a}\langle i,j\rangle}$ is of species *true* and all other level indicators are of species *false*. Figure 2 depicts the chain of level indicators associated with node $\nu_{i,j}$.

LEMMA 15. *For each* $\mathfrak{p} \in \mathfrak{P}$*, for each* $\mathfrak{a} \in \mathfrak{A}$*,*

if $\mathfrak{a}$ *solves* $\mathfrak{p}$ *then* $\mathcal{I}_{\mathfrak{a}}$ *models* $\#(\mathfrak{p})$ *in* $\widehat{\Sigma}$*.*

Proof. For each $\mathfrak{a} \in \mathfrak{A}$, for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$,

$\nu_{i,j} \in \mathcal{D}_{\mathcal{I}_{\mathfrak{a}}}(\text{:file rank} \approx \text{:rank file})$,
$\nu_{i,j} \in \mathcal{D}_{\mathcal{I}_{\mathfrak{a}}}(\text{:level next}^{\mathfrak{a}\langle i,j\rangle} \sim \mathit{true})$,
for each $k \in \mathbb{N}$, if $k \neq \mathfrak{a}\langle i, j\rangle$ then $\nu_{i,j} \in \mathcal{D}_{\mathcal{I}_{\mathfrak{a}}}(\text{:level next}^{k} \sim \mathit{false})$,
$\nu_{i,j} \in \mathcal{D}_{\mathcal{I}_{\mathfrak{a}}}([\text{:level next}^{\mathfrak{a}\langle i,j\rangle} \sim \mathit{true} \ \wedge\ \text{:file level next}^{\mathfrak{a}\langle i+1,j\rangle} \sim \mathit{true}])$, and
$\nu_{i,j} \in \mathcal{D}_{\mathcal{I}_{\mathfrak{a}}}([\text{:level next}^{\mathfrak{a}\langle i,j\rangle} \sim \mathit{true} \ \wedge\ \text{:rank level next}^{\mathfrak{a}\langle i,j+1\rangle} \sim \mathit{true}])$.

Thus, for each $n \in \mathbb{N}$, for each $\mathfrak{p} = \langle \mathfrak{t}_0, \ldots, \mathfrak{t}_n\rangle \in \mathfrak{P}$, for each $\mathfrak{a} \in \mathfrak{A}$,

$\mathfrak{a}$ solves $\mathfrak{p}$
$\Longrightarrow$ for each $\nu \in \mathcal{N}_{\mathfrak{a}}$, $\nu \in \mathcal{D}_{\mathcal{I}_{\mathfrak{a}}}([\![\#_0(\mathfrak{p}) \ \wedge\ \#_1(\mathfrak{p}) \ \wedge\ \#_2(\mathfrak{p}) \ \wedge\ \#_3(\mathfrak{p}) \ \wedge\ \#_4(\mathfrak{p})]\!])$
$\Longrightarrow$ for each $o \in \mathcal{U}_{\mathfrak{a}}$, $o \in \mathcal{D}_{\mathcal{I}_{\mathfrak{a}}}(\#(\mathfrak{p}))$
$\Longrightarrow$ $\mathcal{I}_{\mathfrak{a}}$ models $\#(\mathfrak{p})$ in $\widehat{\Sigma}$.

□

LEMMA 16. *For each* $\mathfrak{p} \in \mathfrak{P}$*, for each nontrivial interpretation* $\mathcal{I}$ *of* $\widehat{\Sigma}$*,*

if $\mathcal{I}$ *models* $\#(\mathfrak{p})$ *in* $\widehat{\Sigma}$ *then* $\mathfrak{p}$ *is soluble.*

Proof. For each nontrivial interpretation $\mathcal{I} = \langle \mathcal{U}, \mathcal{S}, \mathcal{F}\rangle$ of $\widehat{\Sigma}$, for each $n \in \mathbb{N}$, for each $\mathfrak{p} = \langle \mathfrak{t}_0, \ldots, \mathfrak{t}_n\rangle \in \mathfrak{P}$,

$\mathcal{I}$ models $\#(\mathfrak{p})$ in $\widehat{\Sigma}$
$\Longrightarrow$ $\mathcal{I}$ models $\#(\mathfrak{p})$ in $\widehat{\Sigma}$, and
for some $o \in \mathcal{U}$,
$\mathcal{S}(o) = \mathit{node}$ or $\mathcal{S}(o) = \mathit{true}$ or $\mathcal{S}(o) = \mathit{false}$
$\Longrightarrow$ $\mathcal{I}$ models $\#(\mathfrak{p})$ in $\widehat{\Sigma}$, and
for some $o \in \mathcal{U}$,
$\mathcal{S}(o) = \mathit{node}$ or ($\mathcal{F}(\text{earth})(o)$ is defined and $\mathcal{S}(\mathcal{F}(\text{earth})(o)) = \mathit{node}$)
$\Longrightarrow$ $\mathcal{I}$ models $\#(\mathfrak{p})$ in $\widehat{\Sigma}$ and for some $o \in \mathcal{U}$, $\mathcal{S}(o) = \mathit{node}$

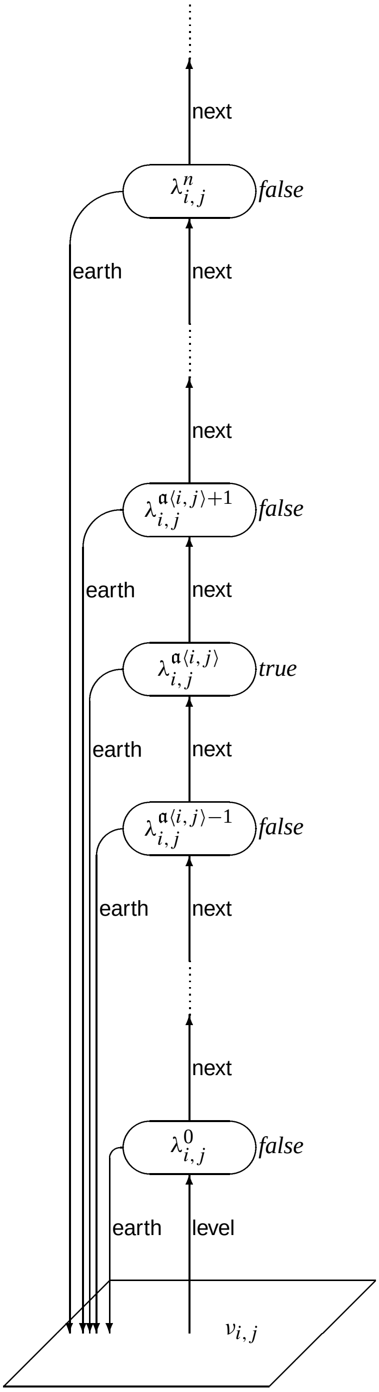

Figure 2. Chain of level indicators associated with a node.

$\Longrightarrow$ $\mathscr{I}$ models $\#(\mathfrak{p})$ in $\widehat{\Sigma}$, and
for some $o \in \mathcal{U}$, for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$,
$\mathcal{T}_{\mathscr{I}}(\textsf{:file}^i\ \textsf{rank}^j)(o)$ is defined and $\mathscr{S}(\mathcal{T}_{\mathscr{I}}(\textsf{:file}^i\ \textsf{rank}^j)(o)) = \textit{node}$
$\Longrightarrow$ for some $o \in \mathcal{U}$, for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$,
$\mathcal{T}_{\mathscr{I}}(\textsf{:file}^i\ \textsf{rank}^j)(o)$ is defined, and
$\mathcal{T}_{\mathscr{I}}(\textsf{:file}^i\ \textsf{rank}^j)(o) \in \mathscr{D}_{\mathscr{I}}([\![\#_0(\mathfrak{p}) \ \wedge\ \#_1(\mathfrak{p}) \ \wedge\ \#_2(\mathfrak{p}) \ \wedge\ \#_3(\mathfrak{p}) \ \wedge\ \#_4(\mathfrak{p})]\!])$
$\Longrightarrow$ for some $o \in \mathcal{U}$, for each $i \in \mathbb{N}$, for each $j \in \mathbb{N}$,
$o \in \mathscr{D}_{\mathscr{I}}(\textsf{:file}^{i+1}\ \textsf{rank}^j \approx \textsf{:file}^i\ \textsf{rank}^j\ \textsf{file})$,
for some unique $k \in \{0, \ldots, n\}$, $o \in \mathscr{D}_{\mathscr{I}}(\textsf{:file}^i\ \textsf{rank}^j\ \textsf{level}\ \textsf{next}^k \sim \textit{true})$,
for some $k \in \{0, \ldots, n\}$, for some $l \in \{0, \ldots, n\}$,
$o \in \mathscr{D}_{\mathscr{I}}(\textsf{:file}^i\ \textsf{rank}^j\ \textsf{level}\ \textsf{next}^k \sim \textit{true})$,
$o \in \mathscr{D}_{\mathscr{I}}(\textsf{:file}^{i+1}\ \textsf{rank}^j\ \textsf{level}\ \textsf{next}^l \sim \textit{true})$, and
$\mathfrak{Right}(\mathfrak{t}_k) = \mathfrak{Left}(\mathfrak{t}_l)$, and
for some $k \in \{0, \ldots, n\}$, for some $l \in \{0, \ldots, n\}$,
$o \in \mathscr{D}_{\mathscr{I}}(\textsf{:file}^i\ \textsf{rank}^j\ \textsf{level}\ \textsf{next}^k \sim \textit{true})$,
$o \in \mathscr{D}_{\mathscr{I}}(\textsf{:file}^i\ \textsf{rank}^{j+1}\ \textsf{level}\ \textsf{next}^l \sim \textit{true})$, and
$\mathfrak{Top}(\mathfrak{t}_k) = \mathfrak{Bottom}(\mathfrak{t}_l)$
$\Longrightarrow$ for some $\mathfrak{a} \in \mathfrak{A}$, $\mathfrak{a}$ solves $\mathfrak{p}$ let $\mathfrak{a}\langle i, j\rangle$ be the unique $k \in \{0, \ldots, n\}$
such that $o \in \mathscr{D}_{\mathscr{I}}(\textsf{:file}^i\ \textsf{rank}^j\ \textsf{level}\ \textsf{next}^k \sim \textit{true})$
$\Longrightarrow$ $\mathfrak{p}$ is soluble.

□

PROPOSITION 17. *For each* $\mathfrak{p} \in \mathfrak{P}$,

$\mathfrak{p}$ *is soluble iff* $\#(\mathfrak{p})$ *is modellable in* $\widehat{\Sigma}$.

Proof. By Lemmata 14, 15 and 16. □

By Proposition 17, # effectively reduces the Π^0_1-hard problem of determining whether an arbitrary finite and nonempty set of domino types can tile $\mathbb{N}^2$ to the problem of determining whether an arbitrary description in $\widehat{\Sigma}$ is modellable in $\widehat{\Sigma}$. Consequently, modellability in $\widehat{\Sigma}$, a finite signature, is Π^0_1-hard.

THEOREM 18. *For some finite signature* Σ,

$\{\delta \in \mathbb{D}_\Sigma \mid \delta$ *is modellable in* $\Sigma\}$ *is* Π^0_1*-hard.*

Proof. By Propositions 13 and 17,

$$\{\mathfrak{p} \in \mathfrak{P} \mid \mathfrak{p} \text{ is soluble}\} \text{ is reducible to } \{\delta \in \mathbb{D}_{\widehat{\Sigma}} \mid \delta \text{ is modellable in } \widehat{\Sigma}\}.$$

Thus, by Proposition 11,

$$\{\delta \in \mathbb{D}_{\widehat{\Sigma}} \mid \delta \text{ is modellable in } \widehat{\Sigma}\} \text{ is } \Pi^0_1\text{-hard.}$$

Thus, by Proposition 12, for some finite signature Σ,

$$\{\delta \in \mathbb{D}_{\Sigma} \mid \delta \text{ is modellable in } \Sigma\} \text{ is } \Pi^0_1\text{-hard.}$$

□

4.3. MODELLABILITY IN SOME FINITE SIGNATURE IS DECIDABLE

Theorem 18 shows that modellability in some, but not necessarily all, finite signatures is Π^0_1-hard. We now show that the existential quantification of Theorem 18 is essential by showing that modellability in some finite signature is decidable, and hence not Π^0_1-hard. For each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$,

Σ is **terminating**
iff for each $n \in \mathbb{N}$,
  if for some $\{\sigma_0, \ldots, \sigma_n\} \subseteq \mathbb{S}$, for some $\{\varphi_1, \ldots, \varphi_n\} \subseteq \mathbb{F}$,
    $\sigma_0 = \sigma_n$ and for each $m \in \{1, \ldots, n\}$, $\sigma_m \in \mathbb{A}\langle \sigma_{m-1}, \varphi_m \rangle$
  then $n = 0$.

We show our desired result by proving that modellability in each finite and terminating signature is decidable. Our proof uses the morphs and admission relation of King (1994). For each signature Σ,

μ is a **morph** in Σ
iff μ is a triple $\langle \beta, \varrho, \lambda \rangle$,
  $\beta \subseteq \mathbb{T}_\Sigma$,
  $: \in \beta$,
  for each $\tau \in \mathbb{T}_\Sigma$, for each $\varphi \in \mathbb{F}$, if $\tau\varphi \in \beta$ then $\tau \in \beta$,
  ϱ is an equivalence relation over β,
  for each $\tau_1 \in \mathbb{T}_\Sigma$, for each $\tau_2 \in \mathbb{T}_\Sigma$, for each $\varphi \in \mathbb{F}$,
    if $\tau_1\varphi \in \beta$ and $\langle \tau_1, \tau_2 \rangle \in \varrho$ then $\langle \tau_1\varphi, \tau_2\varphi \rangle \in \varrho$,
  λ is a total function from β to $\mathbb{S}$,

for each $\tau_1 \in \mathbb{T}_\Sigma$, for each $\tau_2 \in \mathbb{T}_\Sigma$,
if $\langle \tau_1, \tau_2 \rangle \in \varrho$ then $\lambda(\tau_1) = \lambda(\tau_2)$,
for each $\tau \in \mathbb{T}_\Sigma$, for each $\varphi \in \mathbb{F}$,
if $\tau\varphi \in \beta$ then $\lambda(\tau\varphi) \in \mathbb{A}\langle \lambda(\tau), \varphi \rangle$, and
for each $\tau \in \beta$, for each $\varphi \in \mathbb{F}$,
if $\mathbb{A}\langle \lambda(\tau), \varphi \rangle \neq \emptyset$ then $\tau\varphi \in \beta$.

For each signature Σ, we write $\mathbb{M}_\Sigma$ for the set of morphs in Σ.

PROPOSITION 19. *For each finite and terminating signature* Σ*,*

$\mathbb{M}_\Sigma$ *is finite and for each* $\mu = \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma$*,* β*,* ϱ *and* λ *are finite.*

Proof. Suppose that for each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$,

for each $\{\varphi_1 \ldots \varphi_n\} \subseteq \mathbb{F}$,
$:\varphi_1 \ldots \varphi_n$ is legitimate in Σ
iff for some $\{\sigma_0, \ldots, \sigma_n\} \subseteq \mathbb{S}$, for each $m \in \{1, \ldots, n\}$, $\sigma_m \in \mathbb{A}\langle \sigma_{m-1}, \varphi_m \rangle$, and
$\mathbb{L}_\Sigma = \{\tau \in \mathbb{T}_\Sigma \mid \tau \text{ is legitimate in } \Sigma\}$.

Then for each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$, for each $\mu = \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma$,

$\beta \subseteq \mathbb{L}_\Sigma$,
$\varrho \subseteq \mathbb{L}_\Sigma \times \mathbb{L}_\Sigma$, and
$\lambda \subseteq \mathbb{L}_\Sigma \times \mathbb{S}$.

Thus, for each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$,

Σ is finite and terminating
$\Longrightarrow$ $\mathbb{S}$ is finite and $\mathbb{L}_\Sigma$ is finite
$\Longrightarrow$ $\mathbb{M}_\Sigma$ is finite and for each $\mu = \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma$, β, ϱ and λ are finite.

□

For each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$,

for each $\tau \in \mathbb{T}_\Sigma$, $\tau + : = \tau$,
for each $\tau_1 \in \mathbb{T}_\Sigma$, for each $\tau_2 \in \mathbb{T}_\Sigma$, for each $\varphi \in \mathbb{F}$,
$\tau_1 + \tau_2\varphi = (\tau_1 + \tau_2)\varphi$, and

for each $\mu = \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma$, for each $\tau \in \mathbb{T}_\Sigma$,

$\beta/\tau = \{\tau' \in \mathbb{T}_\Sigma \mid \tau + \tau' \in \beta\}$,

$\varrho/\tau = \{\langle \tau_1, \tau_2 \rangle \in \mathbb{T}_\Sigma \times \mathbb{T}_\Sigma \mid \langle \tau + \tau_1, \tau + \tau_2 \rangle \in \varrho\}$,

$\lambda/\tau = \{\langle \tau', \sigma \rangle \in \mathbb{T}_\Sigma \times \mathbb{S} \mid \langle \tau + \tau', \sigma \rangle \in \lambda\}$, and

$\mu/\tau = \langle \beta/\tau, \varrho/\tau, \lambda/\tau \rangle$.

PROPOSITION 20. *For each signature* Σ, *for each* $\mu \in \mathbb{M}_\Sigma$, *for each* $\tau \in \beta$, $\mu/\tau \in \mathbb{M}_\Sigma$.

For each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$, $\mathcal{M}_\Sigma$ is the total function from $\mathbb{D}_\Sigma$ to *Pow*($\mathbb{M}_\Sigma$) such that

for each $\tau \in \mathbb{T}_\Sigma$, for each $\sigma \in \mathbb{S}$, $\mathcal{M}_\Sigma(\tau \sim \sigma) = \left\{ \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma \middle| \langle \tau, \sigma \rangle \in \lambda \right\}$,

for each $\tau_1 \in \mathbb{T}_\Sigma$, for each $\tau_2 \in \mathbb{T}_\Sigma$,

$\mathcal{M}_\Sigma(\tau_1 \approx \tau_2) = \left\{ \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma \middle| \langle \tau_1, \tau_2 \rangle \in \varrho \right\}$,

for each $\delta \in \mathbb{D}_\Sigma$, $\mathcal{M}_\Sigma(\neg\delta) = \mathbb{M}_\Sigma \setminus \mathcal{M}_\Sigma(\delta)$,

for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$, $\mathcal{M}_\Sigma([\delta_1 \wedge \delta_2]) = \mathcal{M}_\Sigma(\delta_1) \cap \mathcal{M}_\Sigma(\delta_2)$,

for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$, $\mathcal{M}_\Sigma([\delta_1 \vee \delta_2]) = \mathcal{M}_\Sigma(\delta_1) \cup \mathcal{M}_\Sigma(\delta_2)$, and

for each $\delta_1 \in \mathbb{D}_\Sigma$, for each $\delta_2 \in \mathbb{D}_\Sigma$,

$\mathcal{M}_\Sigma([\delta_1 \rightarrow \delta_2]) = (\mathbb{M}_\Sigma \setminus \mathcal{M}_\Sigma(\delta_1)) \cup \mathcal{M}_\Sigma(\delta_2)$.

For each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$, for each $\delta \in \mathbb{D}_\Sigma$, for each $\mu = \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma$,

μ **satisfies** δ in Σ iff $\mu \in \mathcal{M}_\Sigma(\delta)$, and

δ **admits** μ in Σ iff for each $\tau \in \beta$, $\mu/\tau \in \mathcal{M}_\Sigma(\delta)$.

PROPOSITION 21. *For each signature* Σ, *for each* $\delta \in \mathbb{D}_\Sigma$,

δ *is modellable in* Σ *iff for some* $\mu \in \mathbb{M}_\Sigma$, δ *admits* μ *in* Σ.

Proof. Firstly, to prove from left to right, we construct for each object in each interpretation a corresponding morph such that a description is true of an object iff the morph corresponding to the object satisfies the description. Thus, if a non-trivial interpretation models a description then the description admits the morph corresponding to some object in the interpretation. Formally, suppose that for each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$, for each interpretation $\mathcal{I} = \langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ of Σ, for each $o \in \mathcal{U}$,

$$\beta_{\mathcal{I}}^{o} = \left\{ \tau \in \mathbb{T}_\Sigma \middle| \mathcal{T}_{\mathcal{I}}(\tau)(o) \text{ is defined} \right\},$$

$$\varrho^o_{\mathcal{I}} = \left\{ \langle \tau_1, \tau_2 \rangle \in \mathbb{T}_\Sigma \times \mathbb{T}_\Sigma \;\middle|\; \begin{array}{l} \mathcal{T}_{\mathcal{I}}(\tau_1)(o) \text{ is defined,} \\ \mathcal{T}_{\mathcal{I}}(\tau_2)(o) \text{ is defined, and} \\ \mathcal{T}_{\mathcal{I}}(\tau_1)(o) = \mathcal{T}_{\mathcal{I}}(\tau_2)(o) \end{array} \right\},$$

$$\lambda^o_{\mathcal{I}} = \left\{ \langle \tau, \sigma \rangle \in \mathbb{T}_\Sigma \times \mathbb{S} \;\middle|\; \begin{array}{l} \mathcal{T}_{\mathcal{I}}(\tau)(o) \text{ is defined, and} \\ \mathcal{S}(\mathcal{T}_{\mathcal{I}}(\tau)(o)) = \sigma \end{array} \right\}, \text{ and}$$

$$\mu^o_{\mathcal{I}} = \langle \beta^o_{\mathcal{I}}, \varrho^o_{\mathcal{I}}, \lambda^o_{\mathcal{I}} \rangle.$$

Then for each signature Σ, for each interpretation $\mathcal{I} = \langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ of Σ, for each $o \in \mathcal{U}$,

$\mu^o_{\mathcal{I}} \in \mathbb{M}_\Sigma$, and

for each $\delta \in \mathbb{D}_\Sigma$,

$o \in \mathcal{D}_{\mathcal{I}}(\delta)$ iff $\mu^o_{\mathcal{I}} \in \mathcal{M}_\Sigma(\delta)$. by arithmetic induction on the complexity of δ

Thus, for each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$, for each $\delta \in \mathbb{D}_\Sigma$,

δ is modellable in Σ

$\Longrightarrow$ for some nontrivial interpretation $\mathcal{I} = \langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ of Σ, $\mathcal{I}$ models δ in Σ

$\Longrightarrow$ for some interpretation $\mathcal{I} = \langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ of Σ, for some $o \in \mathcal{U}$,

for each $\tau \in \mathbb{T}_\Sigma$,

$\tau \in \beta^o_{\mathcal{I}}$

$\Longrightarrow \mathcal{T}_{\mathcal{I}}(\tau)(o)$ is defined and $\mu^o_{\mathcal{I}}/\tau = \mu_{\mathcal{I}}^{\mathcal{T}_{\mathcal{I}}(\tau)(o)}$

$\Longrightarrow \mu^o_{\mathcal{I}}/\tau \in \mathcal{M}_\Sigma(\delta)$

$\Longrightarrow$ for some interpretation $\mathcal{I} = \langle \mathcal{U}, \mathcal{S}, \mathcal{F} \rangle$ of Σ, for some $o \in \mathcal{U}$,

δ admits $\mu^o_{\mathcal{I}}$ in Σ

$\Longrightarrow$ for some $\mu \in \mathbb{M}_\Sigma$, δ admits μ in Σ.

Secondly, to prove from right to left, we construct for each morph a corresponding nontrivial interpretation such that a description admits a morph iff the interpretation corresponding to the morph models the description. Formally, suppose that for each signature $\Sigma = \langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$, for each $\mu = \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma$,

for each $\tau \in \beta$, $[\tau]_\mu$ is the equivalence class of τ in ϱ,

$\mathcal{U}_\mu = \{[\tau]_\mu \mid \tau \in \beta\}$,

$\mathcal{S}_\mu$ is the total function from $\mathcal{U}_\mu$ to $\mathbb{S}$ such that for each $\tau \in \beta$,

$\mathcal{S}_\mu([\tau]_\mu) = \lambda(\tau)$,

$\mathcal{F}_\mu$ is the total function from $\mathbb{F}$ to the set of partial functions from $\mathcal{U}_\mu$ to $\mathcal{U}_\mu$ such that for each $\varphi \in \mathbb{F}$, for each $\tau \in \beta$,

$\mathcal{F}_\mu(\varphi)([\tau]_\mu)$ is defined iff $\tau\varphi \in \beta$, and

if $\mathcal{F}_\mu(\varphi)([\tau]_\mu)$ is defined then $\mathcal{F}_\mu(\varphi)([\tau]_\mu) = [\tau\varphi]_\mu$, and

$\mathcal{I}_\mu = \langle \mathcal{U}_\mu, \mathcal{S}_\mu, \mathcal{F}_\mu \rangle$.

Then for each signature Σ, for each $\mu = \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma$,

$\mathcal{I}_\mu$ is a nontrivial interpretation of Σ, and

for each $\delta \in \mathbb{D}_\Sigma$, for each $\tau \in \beta$,

$\mu/\tau \in \mathcal{M}_\Sigma(\delta)$ iff $[\tau]_\mu \in \mathcal{D}_{\mathcal{I}_\mu}(\delta)$. by arithmetic induction on the complexity of δ

Thus, for each signature Σ, for each $\delta \in \mathbb{D}_\Sigma$,

for some $\mu \in \mathbb{M}_\Sigma$, δ admits μ in Σ

$\Longrightarrow$ for some $\mu = \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma$, for each $\tau \in \beta$, $\mu/\tau \in \mathcal{M}_\Sigma(\delta)$

$\Longrightarrow$ for some $\mu = \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma$, for each $\tau \in \beta$, $[\tau]_\mu \in \mathcal{D}_{\mathcal{I}_\mu}(\delta)$

$\Longrightarrow$ $\mathcal{I}_\mu$ models δ in Σ

$\Longrightarrow$ for some nontrivial interpretation $\mathcal{I}$ of Σ, $\mathcal{I}$ models δ in Σ

$\Longrightarrow$ δ is modellable in Σ.

□

PROPOSITION 21. *For each finite and terminating signature* Σ,

$\{\delta \in \mathbb{D}_\Sigma \mid \delta$ *is modellable in* $\Sigma\}$ *is decidable.*

Proof. For each signature Σ,

Σ is finite and terminating

$\Longrightarrow$ $\mathbb{M}_\Sigma$ is finite and for each $\mu = \langle \beta, \varrho, \lambda \rangle \in \mathbb{M}_\Sigma$, β, ϱ and λ are finite by Proposition 19

$\Longrightarrow$ $\{\delta \in \mathbb{D}_\Sigma \mid$ for some $\mu \in \mathbb{M}_\Sigma$, δ admits μ in $\Sigma\}$ is decidable

$\Longrightarrow$ $\{\delta \in \mathbb{D}_\Sigma \mid \delta$ is modellable in $\Sigma\}$ is decidable. by Proposition 21

□

We immediately have that modellability in some finite signature is decidable.

THEOREM 23. *For some finite signature* Σ,

$$\{\delta \in \mathbb{D}_\Sigma \mid \delta \textit{ is modellable in } \Sigma\} \textit{ is decidable.}$$

Proof. $\langle \emptyset, \emptyset, \emptyset \rangle$ is a finite and terminating signature. Thus, for some finite signature Σ,

$$\{\delta \in \mathbb{D}_\Sigma \mid \delta \text{ is modellable in } \Sigma\} \text{ is decidable.} \qquad \text{by Proposition 22}$$

□

5. Conclusions

In the previous section we proved

- Theorem 10: modellability in each computable signature is Π^0_1,
- Theorem 18: modellability in some finite signature is Π^0_1-hard, and
- Theorem 23: modellability in some finite signature is decidable.

Since each finite signature is a computable signature, we can draw two conclusions. Firstly, Π^0_1-completeness is a least upper bound on the complexity of modellability in both finite signatures and computable signatures, since modellability in all computable signatures (hence all finite signatures) is Π^0_1, and modellability in some finite signature (hence some computable signature) is Π^0_1-hard. Secondly, Π^0_1-completeness is not a lower bound on the complexity of modellability in either finite signatures or computable signatures, since modellability in some finite signature (hence some computable signature) is decidable, hence not Π^0_1-hard. This last observation raises an interesting question. Is there a syntactic characterisation of all – or at least some – of those finite or computable signatures in which modellability is decidable? Proposition 22 shows that modellability is decidable in all finite and terminating signatures, and this is certainly a syntactic characterisation. However, this result is of very limited linguistic use, since no linguistically useful signature we know of is terminating. For example, simplifying slightly, in the SRL rendition $\langle \mathbb{S}, \mathbb{F}, \mathbb{A} \rangle$ of the HPSG signature of Pollard and Sag (1994),

$$\textit{headed-structure} \in \mathbb{A}\langle \textit{phrase}, \mathsf{daughters} \rangle, \text{ and}$$
$$\textit{phrase} \in \mathbb{A}\langle \textit{headed-structure}, \mathsf{head\text{-}dtr} \rangle.$$

Whether there exist syntactic characterisations of linguistically useful finite or computable signatures in which modellability is decidable is an open question for future research.

Acknowledgments

During parts of the writing of this paper, the Deutsche Forschungsgemeinschaft supported King with a postdoctoral stipend at the Graduiertenkolleg Integriertes Linguistikstudium, Seminar für Sprachwissenschaft, Eberhard-Karls-Universität, Tübingen, and the Internationales Zentrum, Eberhard-Karls-Universität, Tübingen supported Simov with a visiting scholarship at the Seminar für Sprachwissenschaft, Eberhard-Karls-Universität, Tübingen. The authors wish to thank the Seminar für Sprachwissenschaft, the Deutsche Forschungsgemeinschaft and the Internationales Zentrum for their support. The authors also wish to thank Patrick Blackburn, Thilo Götz and two anonymous referees for comments on drafts of this paper, and the King and Simov families for taking good care of us as we wrote this paper.

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